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ON THE APPROXIMATIONS AND BIFURCATIONS OF A DYNAMIC SYSTEM

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By means of an example of a classical problem in flight dynamics we examine the influence of approximation on the structure of the partitioning of the phase space and of the parameter space of a dynamic system. For a qualitative investigation of dynamic systems we can use the transition from the original model to a simplified or piece-wise integrable one, by approximating the characteristics in the equations of motion. Here arises the important question of the admissible deviations of the approximating functions from the real characteristics for the preservation of the necessary closeness between the original and the approximating system The concept of necessary closeness is not unique and is determined by the aims of the investigation. For example, it can be understood as the requirement of retaining for the approximating system the same phase space and parameter space partitioning structure as for the original system [1]. In a general formulation the problem reduces to the question of preserving or losing bifurcations during the transition to the approximating system. The difficulties arisising here are connected with the fact that not all the bifurcations may be kept track of by regular methods, and furthermore, for "fused"

approximating systems (piecewise-analytic ones) there may arise new types of bifurcations for which there is no complete classification as yet. Therefore, a comparative analysis of actual dynamic systems under different approximations is of interest. Below we carry out such an analysis on the basis of an example of a classical problem in flight dynamics [2 - 9]. The choice of this problem was dictated by the fact that in the original system a wide collection of bifurcations is possible (all types of bifurcations of the first degree of structural instability are realized) and by the fact that we have succeeded in establishing strictly the parameter space partitioning structure both for the original system (which had not been done to this time) as well as for the approximating systems. Here differences arise in the partitioning structure of the parameter space and of the phase space, permitting us to evaluate the influence of the approximations on the partitioning structure and to uncover, in particular, the important role played by the "saddle index" [10]. The retention of a quantitative closeness of the characteristics did not prove to be obligatory for the preservation of the qualitative partitioning structure of the system's phase and parameter spaces. The use of the saddle index in the qualitative investigation of "fused" systems is based on the possibility of carrying over the assertions concerning the stability conditions for a separatrix loop and the conditions for the birth of limit cycles from it, to nonanalytic systems, preserving the loop in whose composition the analytic saddle occurs. Theorems 44 – 49 in [10], with appropriate changes of formulations, remain valid for the systems mentioned because the method by which they were established carry over to these systems.

1. We consider the system

$$d\varphi / dt = \rho - \cos\varphi = P, \qquad d\rho / dt = 2\rho\,(\lambda - \mu\rho - \sin\varphi) = Q \qquad (1)$$

for the parameter values $\mu \geqslant 0, \lambda \geqslant 1$. In a cylindrical phase space (on the strip $-\pi \leqslant \varphi \leqslant \pi$, $\rho \geqslant 0$ with edges identified) the equilibrium states are

$$O_1\left(-{}^1/_2\pi,\ 0\right),\quad O_2\left({}^1/_2\pi,\ 0\right),\ O_3\,(\varphi_3,\ \rho_3),\ O_4\,(\varphi_4,\ \rho_4)$$

where

$$\rho_{3,4} = \cos\varphi_{3,4} = \frac{\lambda\mu \pm \sqrt{1+\mu^2-\lambda^2}}{1+\mu^2}$$

and the plus sign before the square root corresponds to the point O_3. In the parameter space the points O_3 and O_4 merge on the curve $1 + \mu^2 - \lambda^2 = 0$ while the points O_4 and O_2 on the straight line $\lambda = 1$. System (1) has two equilibrium states above the curve $1+\mu^2 - \lambda^2 = 0$: O_1 a saddle, and O_2 an unstable node. Below the curve there are four singular points: O_1, O_2, O_3 (a node or a focus) O_4 (a saddle). The fusion of the singular points is the simplest bifurcation of system (1). Other possible bifurcations are connected with the change of stability of the equilibrium state O_3, with the bifurcations of separatrices (separatrices going from a saddle to a saddle), and with the appearance of limit cycles from infinity, from a separatrix loop, from a condensation of trajectories, and from the separatrix of a saddle-node singular point. All these bifurcations may be traced for system (1).

1.1. The equilibrium state O_3 has pure imaginary roots of the characteristic equation for points on the curve $\sigma_3 \equiv (P_\varphi' + Q_\rho')_3 = 0$, $P_\varphi' Q_\rho' - P_\rho' Q_\varphi' > 0$, where for ρ and φ we should use the coordinates of point O_3. The curve $\sigma_3 = 0$ can be represented by the equation

$$\lambda(1-2\mu^2) = 3\mu\sqrt{1+\mu^2-\lambda^2}$$

It starts at the point $\mu = \sqrt{1/5}$, $\lambda = 1$ and ends on the curve $1+\mu^2-\lambda^2 = 0$ which it touches at the point $B\,(\sqrt{1/2}, \sqrt{3/2})$. In passing through the curve $\sigma_3 = 0$ in the direction of increasing μ the unstable focus becomes stable and from it there emerges an unstable limit cycle The first Liapunov index for the points of the curve $\sigma_3 = 0$ has the value

$$\alpha_3 = \frac{\pi\lambda^2(1+4\mu^2)}{3\mu\sqrt{2(1-2\mu^2)}} > 0$$

1.2. Let us trace the change in the qualitative structure and the bifurcations as the point in parameter space moves along the curve $1+\mu^2-\lambda^2 = 0$. To the points on the curve there corresponds a composite singular point arising as a result of the fusion of O_3 and O_4. This is a singular point of the saddle-node type for all points of the curve except the point $\mu = 0, \lambda = 1$ for which three singular points are merged in the phase space, and the point B (a degenrate saddle-node). The qualitative pattern of the partitioning of the phase space into trajectories is determined by the presence or absence of limit cycles girding the phase cylinder and by the location of the separatrices bounding the nodal region of the saddle-node singular point. Figure 1 shows the structures which are realized along the curve as the parameter μ increases.

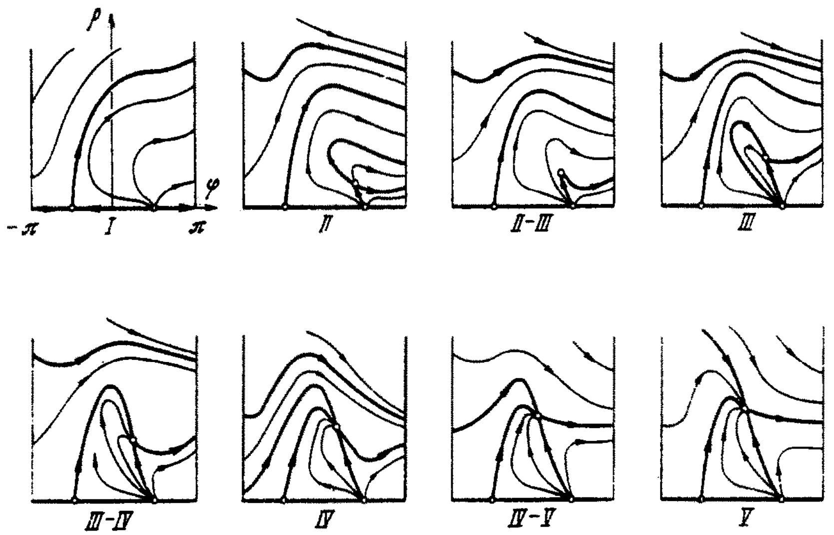

Fig. 1.

The pattern of the phase-cylinder's partitioning into trajectories is shown in Fig. 1(I) for the point A (0, 1) There are no limit cycles (this is seen from the disposition of the contact curve of the system being considered and of the conservative system $\mu = \lambda = 0$). There are only two singular points: the saddle O_1 and the composite singular point O_{234} ($^1/_2\,\pi$, 0). The partitioning structure shown in Fig. 1 (II) is realized on the piece

AB of the curve. Two bifurcations are realized in passing from the point A to the points of piece AB: (1) a singular point of the saddle-node type with an unstable nodal region is separated out from the composite singular point because on the piece AB we have $\sigma_{34} \equiv (P_\varphi' + Q_\rho')_{34} = (1 - 2\mu^2)/\lambda > 0$; (2) from infinity there emerges a stable limit cycle because in the equation there appears the term $-\mu\rho$ and the point at infinity becomes unstable. A bifurcation takes place at point B: the point becomes degenerate and the nodal region disappears. The external manifestation of this is the vanishing of the quantity σ_{34}. When passing through the point B along the curve in the direction of increasing μ the saddle-node singular point with an unstable nodal region turns into a saddle-node with a stable nodal region because the quantity σ_{34} changes sign and becomes negative. The qualitative phase space structure shown on Fig. 1 (III) exists on a certain piece of the curve, abutting point B from the right.

In order to trace the subsequent bifurcations along the curve $1 + \mu^2 - \lambda^2 = 0$ it is essential to determine the qualitative structure of the partitioning into trajectories for large μ and λ. We can show that for large μ and λ the qualitative structure is as shown in Fig. 1 (V). The ω-separatrix of the saddle-node has a negative slope everywhere. There are no limit cycles. (*). Figure 1 represents the qualitative structures successively passing one to the other as the parameters increase along the curve. Here, the ω-separatrix of the saddle-node passes through the structurally unstable [noncoarse] positions shown in Fig. 1 (III-IV) and 1 (IV-V). In Fig. 1 (III-IV) the ω-separatrix of the saddle-node proceeds to the saddle O_1. In Fig. 1 (IV-V) the α- and ω-separatrices of the saddle-node coincide, forming a closed contour girding the cylinder. As the loop originates the stable limit cycle shrinks to it (because for the saddle-node we have $\sigma_{34} < 0$ on the piece of the curve to the right of point B [10]).

1.3. Let us trace the change in qualitative structures and the bifurcations as μ increases along a straight line corresponding to some fixed value of λ from the interval $1 < \lambda < \sqrt{3/2}$ (the straight line is located below point B). The sequence of structures as μ increases is shown in Fig. 2. The pattern of the phase space partitioning into trajectories for $\mu = 0$ is shown in Fig. 2 (1). There are no limit cycles and only two singular points: O_1 is a saddle, O_2 is an unstable node. For a sufficiently small change in μ the number and the nature of the singular points do not change, but the phase space structure does change as a whole. The term $-\mu\rho$ appears in the equation and the point at infinity becomes unstable. A stable limit cycle emerges from infinity. This structure is shown in Fig. 2 (2). As the parameter μ increases the point in the parameter space hits on the curve $1 + \mu^2 - \lambda^2 = 0$ and from the condensation of the trajectories arises a composite saddle-node singular point with an unstable nodal region shown in Fig. 2 (2-3). With a further increase of μ the composite singular point splits up into two simple ones: a saddle and an unstable node (Fig. 2 (3)).

The next bifurcation is traced as the point passes through the curve $\sigma_3 = 0$; here, as μ increases an unstable limit cycle emerges from the equilibrium state. To the bifurcation value of parameter μ there corresponds the trajectory partitioning shown in Fig. 2 (3-4) (the singular point O_3 is a composite focus), while to the values to the right of the curve $\sigma_3 = 0$ (not too far from the curve) there corresponds the pattern shown in Fig. 2 (4). A limit cycle appears around the stable focus.

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*) The proof of this assertion has been given by N. A. Gubar' (Appendix I).

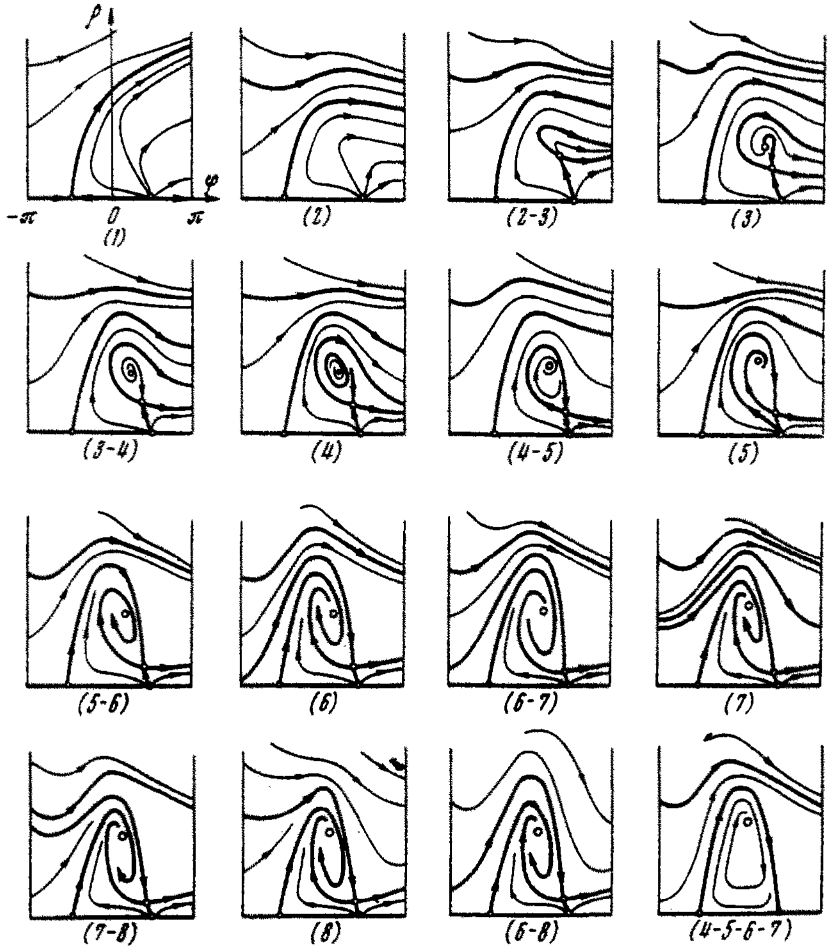

Fig. 2.

Subsequent bifurcations as μ increases are bifurcations of separatrices. Let us trace these bifurcations. On the straight line $\varphi = \arcsin \lambda^{-1}$ located on the strip $-\pi,\ \pi$ between points O_3 and O_4 (the points O_3 and O_4 merge on this straight line if $1 + \mu^2 - \lambda^2 = 0$) we note above the isocline of vertical slopes the points of intersection of the straight line with the three separatrices of saddle O_4 and with the α-separatrix of saddle O_1. If the parameter μ is taken sufficiently close to the curve $\sigma_3 = 0$, then in the order of growth of the coordinate ρ the points are distributed in the following order: P_1 on the ω-separatrix of the saddle, P_2 on the α-separatrix of the saddle, leaving from the saddle to the left, P_3 on the α-separatrix of saddle O_1 and P_4 on the α-separatrix of the saddle, leaving from the saddle to the right. As the parameter μ increases the equilibrium states O_3 and O_4 diverge monotonically along a fixed isocline of vertical slopes ($\partial\varphi_3 / \partial\mu = -\rho_3 (1 + \mu^2 - \lambda^2)^{-1/2} < 0$, $\partial\varphi_4/\partial\mu = \rho_4 (1 + \mu^2 - \lambda^2)^{-1/2} > 0$), while the vector fields on the two sides of the isocline turn in opposite directions: clockwise above and counterclockwise below. The points P_1, P_3 and P_4

lie on separatrices not intersecting the isocline of vertical slopes and, therefore, on the straight line $\varphi = \arcsin \lambda^{-1}$ the point P_1 rises monotonically as μ increases, while the points P_3 and P_4 drop monotonically.

The only possible sequence of bifurcations as μ increases is the one for which the merging of points P_1 and P_2 precedes the merging of P_1, P_3 and P_1, P_4. It is also obvious that if the last of the bifurcations listed is realized, then the rest also are realized. The realizability of the last bifurcation follows from the fact that for sufficiently large μ (when the maximum of the isocline of horizontal slopes, equal to $(\lambda + 1) / \mu$, is less than the maximum of the isocline of vertical slopes, equal to unity) the ω-separatrix of the saddle always has a negative slope and, consequently, the point P_1 lies above the point P_4. It is evident that in this case limit cycles girding the phase cylinder can not exist. The structure of the phase cylinder partitioning into trajectories, shown in Fig. 2 (8), is realized. The arrangements of separatrices, shown in Fig. 2 (5-6) and 2 (6-7), correspond to the merging of the points P_1, P_3 and P_1, P_4. The behavior of the separatrices, to within an even number of limit cycles, determines here the qualitative structure. The values of parameter μ corresponding to the partitionings in Fig. 2 (5-6) and Fig. 2 (6-7) are bifurcation values. As μ varies from these bifurcation values in the increasing or decreasing direction the vector fields on the separatrices turn, respectively, clockwise or counterclockwise, and the separatrices going from a saddle to a saddle are destroyed. The corresponding structurally stable [coarse] structures are shown in Fig. 2 (5), 2 (6) and 2 (7).

We note that although the disposition of separatrices in Fig. 2 (7) also determines the qualitative structure to within an even number of limit cycles, we can assert that there should here exist simultaneously both a stable and an unstable limit cycle girding the cylinder. The unstable limit cycle emerges from the separatrix loop girding the cylinder because the saddle index $\sigma_4 = (P_\varphi + Q'_\omega)_4$ is positive, and when a loop is detroyed, only an unstable limit cycle can shrink to it or appear from it (the curve $\sigma_4 = 0$ represents the equation $\lambda(1 - 2\mu^2) = -3\mu\sqrt{1+\mu^2 - \lambda^2}$; it is located wholly in the strip $\sqrt{3}/2 < \lambda < 3/2$, i.e., outside the interval of variation of λ being considered). With a subsequent increase of parameter μ the vector field on each of the limit cycles turns clockwise; here the stable limit cycle drops out and an unstable one rises. For any fixed λ in the interval being considered there exists a unique bifurcation value μ^* for which the stable and unstable limit cycles merge, forming a double semistable limit cycle. This is the last bifurcation which is possible as the parameter μ grows. With a further increase of μ the vector field on the double cycle turns clockwise and the double limit cycle vanishes. The structurally unstable structure of the partitioning into trajectories, corresponding to the value $\mu = \mu^*$, is shown in Fig. 2 (7-8). Figure 2 (8) shows the partitioning into trajectories realized for all $\mu > \mu^*$.

1.4. The bifurcations as a function of μ for any fixed $\lambda > \sqrt{3}/2$ can be traced in just the same way. Here the number of possible bifurcations lessen but two new ones appear.

1) As μ decreases from the values corresponding to the partitioning shown in Fig. 1 (V) (for the part of the boundary curve above point D in Fig. 3), the saddle-node singular point vanishes and from the α-separatrix of the saddle-node there appears a stable limit cycle (for a reverse variation of μ the stable limit cycle becomes the α-separatrix of the saddle-node).

2) Since the curve $\sigma_4 = 0$ is situated above the straight line $\lambda > \sqrt{3/2}$ and the formation of the separatrix loop for some values of λ can be realized for $\sigma_4 < 0$, then for these values of λ the transition from a partitioning of type (6) in Fig. 2 to type (8), as μ increases, takes place by the shrinking of a stable limit cycle to a separatrix loop girding the cylinder. Here a new structurally unstable structure arises separating structures (6) and (8), shown in Fig. 2 (6-8). For the structure (6-8), as also for the structure (6-7), the α- and ω-separatrices of saddle O_4 form a loop girding the cylinder, but there is no stable limit cycle.

For any λ the structures (1) and (2) of Fig. 2 are realized in the region $1+\mu^2 - \lambda^2 < 0, \mu \geqslant 0$ Structure (8) of Fig. 2 is realized in the region $(1 + \lambda) / \mu < 1$ The change of structure takes place as μ varies in the interval between the curve $1 + \mu^2 - \lambda^2 = 0$ and the straight line $1 + \lambda - \mu = 0$ $(\sqrt{\lambda^2 - 1} < \mu < \lambda + 1)$.

The sets of points corresponding to the structurally unstable bifurcation patterns (4-5), (5-6), (6-7) and (6-8) in Fig. 2 form continuous curves $\{4.5\}, \{5.6\}$, $\{6.7\}$ and $\{6.8\}$ in the $\mu\lambda$-plane. These curves have a positive slope. The latter follows from the fact that as the parameters μ and λ increase individually the vector field on separatrices going from saddle to saddle and not intersecting the contact curve (the isocline of vertical slope) turns in opposite directions. Only for a simultaneous increase or decrease of μ and λ the rotation of a vector field along saddle-to-saddle separatrices can be nonmonotonic and nondestructive of the separatrices. The bifurcation curves $\{4.5\}$ and $\{5.6\}$ start and end on the lines $\lambda = 1$ and $1 + \mu^2 - \lambda^2 = 0$.The curve $\{4.5\}$ does not leave the strip $1 < \lambda < \sqrt{3/2}$ and ends at point B. The separatrix of the composite singular point in Fig. 1 (II-III) can be considered as a degeneracy of the separatrix of point O_4 in Fig. 2 (4-5) under a limit transition preserving the separatrix loop when the points O_3 and O_4 approach. The curve $\{5.6\}$ ends at the point C (Fig. 3). At point C, as on the curve $\{5.6\}$ the separatrix of saddle O_1 goes into saddle O_4 (Fig. 1 (III-IV) and Fig. 2 (5-6)). This is however impossible for any one point of any straight line $\lambda =$ const passing above the point C. The curve $\{6.7\}$ starts on the straight line $\lambda = 1$ and ends on the curve $\sigma_4 = 0$, next it turns into the curve $\{6.8\}$ ending at point D of the curve $1 + \mu^2 - \lambda^2 = 0$. At point D, as on the curves $\{6.7\}$ and $\{6.8\}$, the α- and ω-separatrices of saddle O_4 form a loop. This is impossible for any one point of any straight line $\lambda =$ const passing above the point D. The curves $\{4.5\}$, $\{5.6\}$ and $\{6.7\}$ intersect at one point on the straight line $\lambda = 1$. At this point a trajectory partitioning structure of a high order of structural instability is realized, it is shown in Fig. 2 (4-5-6-7). The point $(1/2\,\pi, 0)$ is a composite singular point. An arbitrarily small variation of the parameters can lead to the structurally unstable structure (4-5), (5-6) or (6-7) only from the structure (4-5-6-7), but because a variation of μ destroys the loop, on the straight line $\lambda = 1$ there can exist only a single point with a structure containing a separatrix loop, namely the point of intersection of the curves $\{4.5.\}$ $\{5.6\}$ and $\{6.7\}$. The set of points corresponding to the bifurcation pattern (7-8) with the double semistable limit cycle in Fig. 2 forms a continuous curve $\{7.8\}$ with a positive slope. The curve $\{7.8\}$ starts on the straight line $\lambda = 1$ and ends at the point of intersection of the curves $\{6.7\}$ and $\{6.8\}$, one being the continuation of the other, with the curve $\sigma_4 = 0$ (point E in Fig. 3).

Figure 3 is a representation, not to scale, of the distribution of the bifurcation curves in the $\mu\lambda$-plane for the case $\mu \geqslant 0$, $\lambda \geqslant 1$ being considered (*).

(*) A distribution of bifurcation curves has been given in [9], where the conclusions on

2. Let us consider system (1) for approximations by the saw-tooth functions

$$\sin\varphi \sim s_2 = \begin{cases} -\dfrac{2}{\pi}\varphi - 2 & \left[-\pi,\ -\dfrac{\pi}{2}\right] \\ \dfrac{2}{\pi}\varphi & \left[-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right], \\ -\dfrac{2}{\pi}\varphi + 2 & \left[\dfrac{\pi}{2},\ \pi\right] \end{cases} \qquad \cos\varphi \sim c_2 = \begin{cases} \dfrac{2}{\pi}\varphi + 1 & [-\pi, 0] \\ -\dfrac{2}{\pi}\varphi + 1 & [0,\ \pi] \end{cases} \tag{2}$$

The equilibrium states on the strip $-\pi \leqslant \varphi \leqslant \pi$ are: $O_1(-{}^1/_2\pi,\ 0)$, $O_2\ ({}^1/_2\pi,\ 0)$, $O_3\ (\varphi_3,\ \rho_3)$, $O_4\ (\varphi_4,\ \rho_4)$, where

$$\varphi_3 = \frac{\pi(\lambda-\mu)}{2(1+\mu)}, \quad \rho_3 = \frac{1+\lambda}{1+\mu}, \quad \varphi_4 = \frac{\pi(\lambda-\mu)}{2(1-\mu)}, \quad \rho_4 = \frac{1-\lambda}{1-\mu}$$

O_1 is a fused saddle, O_2 is a fused unstable node, O_3 is a node or a focus, O_4 is a saddle. In the parameter space the points O_3 and O_4 merge on the straight line $\lambda - \mu = 0$ and the points O_4 and O_2, on the straight line $\lambda = 1$.

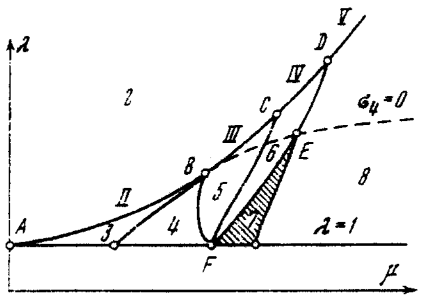

Fig. 3.

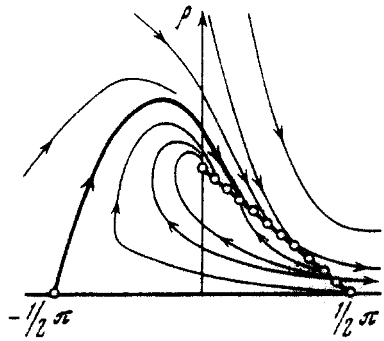

Fig. 4.

The structure of the phase space partitioning for the point $\lambda = \mu = 1$. When $\lambda = \mu = 1$ the isoclines of vertical and horizontal slopes coincide on the interval $0 \leqslant \varphi \leqslant {}^1/_2\pi$ and a phase-space partitioning structure with a rest segment arises on the interval $0 \leqslant \varphi \leqslant {}^1/_2\pi$. Integral curves on which the representative point moves on the interval $0 \leqslant \varphi \leqslant {}^1/_2\pi$ are exponential curves. The rest segment is $\rho = 1 - 2\pi^{-1}\varphi\ (0 \leqslant \varphi \leqslant {}^1/_2\pi)$ stable on the interval $0 < \varphi < {}^1/_2(\pi - 1)$ and unstable on the interval ${}^1/_2(\pi - 1) < \varphi < {}^1/_2\pi$. The integral curve $\rho = \pi^{-1}e^{\pi-1-2\varphi}$ is tangent to the rest segment at the point $({}^1/_2(\pi - 1),\ \pi^{-1})$ and when $\varphi = 0$ falls into the region above the maximum of the isocline of horizontal slopes $(\pi^{-1}e^{\pi-1} > 2)$ and goes off to infinity. There are no limit cycles. All trajectories go to the stable part of the rest segment. The pnase-space partitioning structure in the neighborhood of the rest segment is shown in Fig. 4.

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the boundary-curve partitioning structures and on the birth of a limit cycle from the separatrix loop of a saddle-node were obtained from a graphical construction of the separatrices by the method of isoclines.

The structure of the partitioning on the straight line $\lambda = \mu$. As λ and μ increase from the value $\lambda = \mu = 1$ along a straight line the rest segment splits up and at its end points there arise the singular points: O_{34} $(0, 1)$, being fused from a focus and a saddle, and O_2 $(^1/_2\pi,\ 0)$, being a fused node (unstable). The isocline of horizontal slopes is located above the isocline of vertical slopes on the interval $0 < \varphi < {}^1/_2\pi$ and the separatrix of saddle O_1 ending on the stable piece of the rest segment when $\lambda = \mu = 1$ turns into a trajectory winding into a limit cycle girding the cylinder (the point at infinity is unstable). A stable limit cycle emerges from the trajectory abutting the rest segment and from a piece of the rest segment.

As λ and μ increase along a straight line the saddle-focus O_{34} turns into a saddle-node with a stable nodal region when $\mu = (1 + 2\sqrt{\pi})/\pi = \mu^*$. For μ close to μ^* both the ω-separatrices of the fused saddle-node must leave the node O_2. There are no limit cycles for large λ and μ because the ω-separatrix occurring at the saddle-node has a negative slope everywhere. The validity of this follows from the fact that if we take a point $(\varphi_0,\ \eta_0 > 1)$ on the ω- separatrix, then for sufficiently large μ the coordinate η_0 on the straight line $\varphi = \varphi_0$ is larger than the maximum of the isocline of horizontal slopes $(\lambda + 1) / \mu$, because the vector field turns clockwise in the region $\rho > 1$ as μ increases along the straight line and here η_0 grows, while $(\lambda + 1) / \mu \to 1$.

The qualitative structures, successively passing one to the other as μ and λ increase along the straight line $\lambda = \mu$, are equivalent to some of those shown in Fig. 1. For any μ from the interval $0 < \mu < \mu^*$ the phase-space partitioning structure is equivalent to that shown in Fig. 1 (II-III), for $\mu^* < \mu < \mu_1$, in Fig. 1 (III), for $\mu_1 < \mu < \mu_2$, in Fig. 1 (IV). For $\mu_2 < \mu < \infty$ the distribution of separatrices is shown in Fig. 1 (V).

The structure of the partitioning on the halfline $\mu = \pi(\lambda - 1) + 1 > 1$. As λ and μ increase from the values $\lambda = \mu = 1$ along a halfline, a piece of the isocline on the interval $0 < \varphi < {}^1/_2\pi$ turns around the point $(^1/_2(\pi - 1),\ \pi^{-1})$ and the rest point splits up and gives rise to three singular points: O_3 $(\varphi_3,\ \rho_3)$ a stable focus or node, O_4 $(^1/_2(\pi - 1),\ \pi^{-1})$ a saddle the directions of whose separatrices are determined by the equation $\pi^2k^2 + 2\pi(1 + \mu)k + 4 = 0$ and O_2 $(^1/_2\pi,\ 0)$ a fused node (unstable). As the parameters vary along a straight line the contact curve of the curves of the degenerate system $(\mu = \lambda = 1)$ is $\rho = \pi^{-1}$ and, consequently, always passes through the saddle. The vector field in the region $\rho > 1$ turns clockwise as μ increases and, therefore, the ω- separatrix going into the saddle along the direction $k < -2\pi^{-1}$ cannot intersect the integral curve $\rho = \pi^{-1}e^{\pi-1-2\varphi}$ of the degenrate system, being tangent to the rest segment at just that point at which the saddle arises for $\mu > 1$ and going into the saddle along the direction $k = -2\pi^{-1}$. The separatrix intersects the axis $\varphi = 0$ at the point $\rho^* > \pi^{-1}e^{\pi-1} > 2$ and enters the region above the maximum of the isocline of horizontal slopes. There are no limit cycles girding the cylinder for any values of λ and μ on the halfline being considered. The phase-space partitioning structure for all points of this halfline is the same and is equivalent to the one shown in Fig. 2 (8).

Parameter-space partitioning into regions with different qualitative phase-space structure. Let us observe the change of structures and the bifurcations as μ varies, for a fixed $\lambda = \lambda_0$ from the interval $1 < \lambda < \mu_1$. When $\mu = 0$ the qualitative phase-space partitioning pattern is equivalent to the one in

Fig. 2 (1). The point at infinity is stable. For μ in the interval $0 < \mu < \lambda_0$ the qualitative pattern is equivalent to the one in Fig. 2 (2). The point at infinity is unstable. A stable limit cycle appears from it. When $\mu = \lambda_0$ at the point (0, 1) there appears a fused saddle-focus (if $\mu < \mu^*$) or a fused saddle-node with a stable nodal region (if $\mu > \mu^*$). The qualitative pattern is equivalent to the one in Fig. 1 (II-III) or 1 (III), respectively. As μ increases further the composite fused singular point splits up into simple ones: a saddle O_4 (φ_4, ρ_4) and a stable focus (node O_3 (φ_3, ρ_3). The qualitative pattern is equivalent to the one in Fig. 2 (5). Both the ω-separatrices of saddle O_4 go into the point O_2 (an unstable node). For $\mu \geqslant \pi(\lambda - 1) + 1$ one of the ω-separatrices of saddle O_4 goes off to infinity. The qualitative pattern is equivalent to the one in Fig. 2 (8). As μ increases, two bifurcations of separatrices are realized between the lines $\lambda = \mu$ and $\mu = \pi(\lambda - 1) + 1$: for some $\mu = \mu^{(1)}(\lambda_0)$ a separtrix arises going from saddle O_4 to saddle O_1, and for $\mu = \mu^{(2)}(\lambda_0) > \mu^{(1)}(\lambda_0)$ a separatrix loop girding the cylinder arises.

The nature of the bifurcation as the separatrix loop arises and vanishes is determined by the sign of the saddle index

$$\sigma_4 = (P_\varphi' + Q_\rho')_4 = \frac{2}{\pi(\mu - 1)}(\pi\mu + \mu - \pi\mu\lambda - 1)$$

The curve $\sigma_4 = 0$ is tangent to the straight line $\mu = \pi(\lambda - 1) + 1$ at the point $\lambda = \mu = 1$ and is located to the right of it. To the left of the straight line the saddle index has a negative value for bifurcation values of the parameters. As μ increases, as a separatrix loop arises the stable limit cycle shrinks to the loop. For a fixed $\lambda = \lambda_1$ from the interval $\mu_1 < \lambda < \mu_2$ only one bifurcation is realized as μ increases on the interval $\lambda_1 < \mu < \pi(\lambda_1 - 1) + 1$: a separatrix loop girding the cylinder, to which the stable limit cycle shrinks, appears when $\mu' = \mu^{(2)}(\lambda_1)$. For a fixed $\lambda = \lambda_2 > \mu_2$ no changes take place in the qualitative structures as μ varies on the interval $\lambda_2 < \mu < \infty$ The curves $\mu = \mu^{(1)}(\lambda)$ and $\mu = \mu^{(2)}(\lambda)$, corresponding to structurally unstable structures which are qualitatively equivalent to those in Fig. 2 (5-6) and 2 (6-8), form bifurcation curves, starting at the point $\lambda = \mu = 1$ and ending on the straight line $\lambda = \mu$ at the points $\mu = \mu_1$ and $\mu = \mu_2$ respectively.

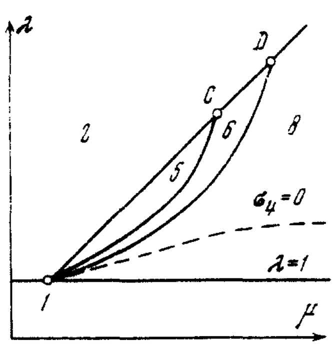

Fig. 5.

The partitioning of the space of the parameters $\mu \geqslant 0$, $\lambda \geqslant 1$ is shown in Fig. 5 (not to scale). The qualitative nature of the parameter-space partitioning differs essentially from the partitioning for the original system (1). In particular, the region wherein two limit cycles girding the phase cylinder exist is absent here.

3. Let us consider system (1) under the approximations

$$\sin\varphi \sim s_3 = \begin{cases} -\frac{2}{\pi}\varphi - 2 \left[-\pi,\ -\frac{\pi}{2}\right] \\ \frac{2}{\pi}\varphi \quad \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right] \\ -\frac{2}{\pi}\varphi + 2\left[\frac{\pi}{2},\ \pi\right] \end{cases}, \quad \cos\varphi \sim c_3 = \begin{cases} \frac{2}{\pi}\varphi + 1\ [-\pi,\ 0] \\ 1 - \frac{4}{\pi^2}\varphi^2 \left[0,\ \frac{\pi}{2}\right] \\ -\frac{2}{\pi}\varphi + 1\left[\frac{\pi}{2},\ \pi\right] \end{cases} \quad (3)$$

differing from (2) in that under the approximation of $\cos\varphi$ a segment of the straight line changes into a parabola on the interval $0,\ {}^1/_2\pi$ Such a change makes it impossible to have a composite bifurcation with a coincidence of isoclines on the segment and essentially changes the total pattern of possible bifurcations. The equilibrium states on the strip $-\pi \leqslant \varphi \leqslant \pi$ are: O_1 $(-{}^1/_2\pi,\ 0)$ a fused saddle, O_2 $({}^1/_2\pi,\ 0)$ a fused unstable node, O_3 $(\varphi_3,\ \rho_3)$ a node or a focus, O_4 $(\varphi_4,\ \rho_4)$ a saddle. Here φ_4 is the larger root of the equation

$$\varphi^2 - \frac{\pi}{2\mu}\varphi + \frac{\pi^2(\lambda-\mu)}{4\mu} = 0 \qquad (0 \leqslant \varphi \leqslant {}^1/_2\pi)$$

while φ_3 either is its smaller root, if $\lambda - \mu > 0$ or is determined by the formula presented at the beginning of Sect. 2, if $\lambda - \mu < 0$. The quantities ρ_4 and ρ_3 are found from the equations of the corresponding isoclines. In the parameter space the points O_3 and O_4 merge on the curve $4\mu^2 - 4\mu\lambda + 1 = 0$ $(\mu > {}^1/_2)$; the points O_4 and O_2 merge on the straight line $\lambda = 1$. In the parameter plane the straight line segment $\lambda = 1,\ 0 < \mu < {}^1/_2$ and a branch of the curve $4\mu^2 - 4\mu\lambda + 1 = 0,\ \ \mu > {}^1/_2$, form the boundary of the region wherein only two points (O_1 and O_2) exist.

Birth of a limit cycle from a focus. The focus O_3 has a change of stability on the curve

$$\sigma_3 \equiv (P_\varphi' + Q_\rho')_3 = \frac{2+\pi}{\mu\pi}\left(1 - \sqrt{1 - 4\mu\lambda + 4\mu^2}\right) - 2\lambda = 0$$

starting at the point $\lambda = 1,\ \mu = (2+\pi)\ /\ (2+2\pi)$ and ending on the curve $4\mu^2 - 4\mu\lambda + 1 = 0$, which it touches at the point B

$$\mu = [(4+\pi)\ /\ 4\pi]^{1/2} = \mu_0, \qquad \lambda = (2+\pi)\ /\ [\pi\,(4+\pi)]^{1/2}$$

In passing through the curve $\sigma_3 = 0$ in the direction of increasing μ the unstable focus becomes stable and from it appears an unstable limit cycle (for points on the curve $\sigma_3 = 0$ the first Liapunov index has the value

$$\sigma_3 = \mu\frac{2+\pi}{\pi}\sqrt{\frac{\mu}{2\pi\lambda}}\ \frac{(2+\pi)^2 + 4\pi^2\lambda\mu}{[2+\pi-2\pi\lambda\mu]^{3/2}} > 0$$

Phase-space partitioning structure on the boundary curve separating the regions of two and of four points. A phase space with a saddle-node singular point, arising from the merging of points O_3 and O_4 corresponds to the points on the curve $4\mu^2 - 4\lambda\mu + 1 = 0$ $(\mu > {}^1/_2)$. When $\mu = \mu_0$ the directions along which the trajectories may enter the singular point coincide, and the saddle-node becomes degenerate. In passing through the value $\mu = \mu_0$ the saddle-node with an unstable nodal region ($\mu < \mu_0$) turns into a saddle-node with a stable nodal region ($\mu > \mu_0$). For small μ the separatrix of saddle O_1 winds onto the limit cycle girding the cylinder. For large μ the ω-separatrix of the saddle-node has a negative slope everywhere (Appendix 2) and, consequently, there are no limit cycles. As μ increases along the curve $4\mu^2 - 4\lambda\mu + 1 = 0$, $\mu > 1/2$, the successive qualitative patterns are as in Fig. 1.

Parameter-space partitioning into regions with different qualitative phase-space structure. The vanishing of the saddle index

$$\sigma_4 \equiv (P_\varphi' + Q_\rho')_4 = \frac{2+\pi}{\mu\pi}\left(1 + \sqrt{1 - 4\lambda\mu + 4\mu^2}\right) - 2\lambda$$

takes place on the curve tangent to the boundary curve at point B and having the asymptote $\lambda = 1 + 2/\pi$. The saddle index is negative above the curve $\sigma_4 = 0$.

Starting off from the known phase-space partitioning structures on the boundary curve, we can trace the change in the qualitative structures as μ increases by an almost-verbatim repetition of the arguments carried out in Sect. 1, Items 1.3 and 1.4 because the rotation of the direction field is monotonic as μ varies, while the adoption of approximations (3) does not essentially alter the behavior of the indices σ_3, α_3 and σ_4, which determine the nature of the bifurcations possible in the neighborhood of a focus and of the separatrix loop. For the approximating system, as for the original system (1), there are no limit cycles when $(\lambda + 1)/\mu < 1$ and all the bifurcations are realized on the segment $\lambda = \text{const}$ between the boundary curve and the straight line $\mu = \lambda + 1$. The qualitative nature of the partitionings of the parameter space and of the phase space under approximations (3) remains the same as for the original system (1).

4. Let us consider system (1) under a piecewise-constant approximation for $\sin\varphi$ and a saw-tooth approximation for $\cos\varphi$

$$\sin\varphi \sim s_4 = \begin{cases} -1 & (-\pi, 0) \\ 1 & (0, \pi) \end{cases}, \qquad \cos\varphi \sim c_4 = \begin{cases} 2\pi^{-1}\varphi + 1 & [-\pi, 0] \\ -2\pi^{-1}\varphi + 1 & [0, \pi] \end{cases} \tag{4}$$

Note that these are integrable approximations. Under approximations (4) the right hand side of system (1) undergoes discontinuities on the lines of fusing. Besides the straight line $\rho = 0$ the role of the isocline of horizontal slopes is played by the polygonal line consisting of pieces of the integral lines $\rho = (\lambda + 1)/\mu = \rho_3\ (-\pi, 0)$, $\rho = (\lambda - 1)/\mu = \rho_4\ (0, \pi)$ and of the segments between them $\varphi = 0, \pm\pi$; $(\lambda - 1)/\mu < \rho < (\lambda + 1)/\mu$, on which the derivative changes sign. When $(\lambda - 1)/\mu > 1$ there are only two equilibrium states on the strip $-\pi < \varphi < \pi$: $O_1\ (-1/2\pi, 0)$ a saddle, and $O_2\ (1/2\pi, 0)$ an unstable node. When $(\lambda - 1)/\mu = 1$ the isoclines come together and a fused composite singular point $(0, 1)$ arises, qualitatively equivalent to a degenerate saddle-node without a nodal region (Fig. 6). When $(\lambda - 1)/\mu < 1 < (\lambda + 1)/\mu$ the composite singular point splits into two: $O_3\ (0, 1)$ a fused focus, and $O_4\ (\varphi_4, \rho_4)$ a saddle. When $(\lambda + 1)/\mu \leqslant 1$ the focus O_3 turns into a stable node $O_3\ (\varphi_3, \rho_3)$. The straight line $\lambda - \mu - 1 = 0$ is the boundary between the regions wherein two points and four points exist. The points O_4 and O_2 merge on the straight line $\lambda = 1$.

Birth of a limit cycle from the fused focus. The fused focus is stable if the indices

$$\alpha_2 \equiv \frac{1}{3}\left[\frac{F_2'''(0,1)}{F_2''(0,1)} - \frac{F_1'''(0,1)}{F_1''(0,1)}\right], \qquad Q_1(0,1) = 2(\lambda - \mu + 1)$$

have different signs. Here $F_1(\varphi, \rho) = C_1$ $(\varphi \leqslant 0)$ and $F_2(\varphi, \rho) = C_2$ $(\varphi \geqslant 0)$ are the total integrals of the system [11]. The fused focus has a change of stability on the curve

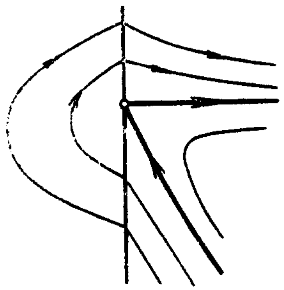

Fig. 6.

$$\alpha_2 = \frac{\pi}{3}\frac{(1+\pi)\mu - \lambda}{(\lambda - \mu + 1)(\lambda - \mu - 1)} = 0$$

starting at the point $\mu = (1 + \pi)^{-1}$, $\lambda = 1$ and ending on the boundary curve at the point $\mu = \pi^{-1}$, $\lambda = (1 + \pi)\pi^{-1}$. As it passes through the curve $\alpha_2 = 0$ in the direction of increasing μ the unstable fused focus becomes stable and an unstable limit cycle appears from it (the index α_4 — an analog of the first Liapunov index [11] — is positive for the points of the bifurcation line $(1 + \pi)\mu - \lambda = 0$).

Parameter-space partitioning into regions with a different qualitative phase-space structure. The nature of the composite singular point and the structure of the phase-space partitioning into trajectories are preserved along the whole boundary curve. Infinity in unstable. The α-separatrix of saddle O_1 cannot go into the singular point and is wound onto a stable limit cycle girding the cylinder. The qualitative pattern of the partitioning into trajectories on the whole boundary curve is equivalent to that in Fig. 1 (II-III). For $(\lambda + 1) / \mu < 1$ the ω-separatrix of saddle O_4, going into the singular point along the direction $k = -2\pi^{-1} - 2\mu$, falls into the region of negative slopes when $\varphi = 0$ and goes off to infinity. There are no limit cycles. The qualitative pattern is equivalent to that in Fig. 2 (8).

The saddle index

$$\sigma_4 = (P_\varphi' + Q_\rho')_4 = 2\pi^{-1}(1 + \pi - \lambda\pi)$$

vanishes on the straight line $\lambda = (1 + \pi) / \pi$ joining up with the line on which the focus O_3 changes stability, at the point of intersection with the boundary curve $\lambda - \mu - 1 = 0$. The saddle index is negative above the straight line $\sigma_4 = 0$. Starting from the known structures of the phase-space partitioning on the boundary curve and in the region $\lambda - \mu + 1 < 0$, once again we can easily trace all the bifurcations and the change in the qualitative structures under a monotonic rotation of the vector field as the parameter μ increases. The succession of qualitative structures, passing one into the other as μ increases, is equivalent to the succession of structurally stable structures (2) - (8) (if $1 < \lambda < 1 + \pi^{-1}$) or of structures (2), (5), (6), (8) (if $1 + \pi^{-1} < \lambda$). The structurally unstable structures separating the structurally stable ones listed are also qualitatively equivalent to the structurally unstable ones shown in Fig. 2, excepting the structure (2-3), and to the structure (2-5) (the latter does not appear in Fig. 2) which should be replaced by the structure (II-III) of Fig. 1 (there is a degenerate saddle-node instead of a saddle-node with an unstable or a stable nodal region). The qualitative parameter-space partitioning structure differs from the partitioning structure for the original system (1) only in that the bifurcation curves $\{5.6\}$ and $\{6.8\}$ do not intersect the boundary curve{2.5}and go off to infinity.

5. By a small change in approximation (4) we can obtain a parameter-space partitioning pattern which coincides qualitatively with the partitioning for the original system (1). We consider system (1) with the approximations

$$\sin\varphi \sim s_5 = \begin{cases} -1(-\pi,\ -\varphi_0] \\ \frac{\varphi}{\varphi_0}[-\varphi_0,\ \varphi_0], \\ +1[\varphi_0,\ \pi) \end{cases} \qquad \cos\varphi \sim c_5 = \begin{cases} \frac{2}{\pi}\varphi + 1\,[-\pi,\ 0] \\ \\ -\frac{2}{\pi}\varphi + 1\,[0,\ \pi] \end{cases} \tag{5}$$

differing from (4) in the approximation of $\sin\varphi$ on the interval $-\varphi_0,\ \varphi_0$. For a small φ_0 approximations (5) are close to (4). The points $O_1\,(-{}^1/_2\pi, 0)$ and $O_2\,({}^1/_2\pi, 0)$ are of the same nature as under approximations (4). The points O_4 and O_2 merge on the straight line $\lambda = 1$. The boundary curve on which the points O_3 and O_4 merge is a polygonal line composed of two links: a segment of the straight line $\lambda = (1 - 2\pi^{-1}\varphi_0)\mu + 1$ for $0 < \mu < {}^1/_2\pi\varphi_0^{-1}$ and a segment of the halfline $\lambda = \mu$ for $\mu > {}^1/_2\pi\varphi_0^{-1}$. The nature of the fused composite singular point and the qualitative structures of the partitioning into trajectories vary along the boundary curve.

If φ_0 is not large, the composite singular point $O_{34}\ (\varphi_0,\ (\lambda - 1)\ /\ \mu)$ on the interval $0 < \mu < {}^1/_2\pi\varphi_0^{-1}$ is made up from the saddle O_4 and the focus or node O_3. For $\mu > {}^1/_2\pi\varphi_0^{-1}$ the composite singular point $\bar{O}_{34}\ (0,\ 1)$ is a saddle-node. For φ_0 satisfying the condition $1 + \pi^{-1} < {}^1/_2\varphi_0^{-1}$ the boundary point $\lambda = \lambda^*$ separating saddle-nodes and saddle-foci lies in the interval $1 + \pi^{-1} < \lambda^* < {}^1/_2\pi\varphi_0^{-1}$. If $1 < \lambda < \lambda^*$ and, consequently, the point $O_{34}(\varphi_0,\ (\lambda - 1)\ /\ \mu)$ is a saddle-focus, the α-separatrix of saddle O_1 cannot go into a singular point and must wind around a stable limit cycle girding the cylinder (infinity is unstable). The qualitative pattern is equivalent to that in Fig. 1 (II-III). For large $\mu = \lambda$ the ω-separatrix of the saddle-node $\bar{O}_{34}\ (0,\ 1)$ has a negative slope everywhere. There are no limit cycles. The qualitative pattern is equivalent to that in Fig. 1 (V). As μ increases along the boundary curve all the relatively structurally stable and the bifurcation structures shown in Fig. 1 from (II-III) to (V) are realized.

Parameter-space partitioning into regions with a different qualitative phase-space structure. The focus $O_3\ (\varphi^*,\ \rho^*)$ is always stable if it is located to the left of the axis $\varphi = 0$ ($\lambda < \mu$), and can have a change in stability if it is located to the right ($\mu < \lambda < (1 - 2\pi^{-1}\varphi_0)\mu + 1$). If $\varphi^* > 0$

$$\sigma_3 = (P_\varphi' + Q_\rho')_3 = \frac{2}{\pi} - 2\lambda - 4\mu\rho^* = \frac{2}{\pi} - 2\mu\rho^* =$$

$$= \frac{2\,[2\varphi_0\pi\lambda\mu - (2\varphi_0 + \pi^2)\,\mu + \pi]}{\pi\,(\pi - 2\varphi_0\mu)}$$

$$\left(\varphi^* = \frac{\pi\varphi_0\,(\lambda - \mu)}{\pi - 2\varphi_0\mu},\quad \rho^* = \frac{\pi - 2\varphi_0\lambda}{\pi - 2\varphi_0\mu}\right)$$

The focus changes stability on a curve starting at the point $\lambda = 1$, $\mu = \pi\,[\pi^2 - (2\pi - 2)\,\varphi_0]^{-1}$, and ending on the boundary curve at the point where $\lambda = 1 + \pi^{-1}$. In passing through the curve $\sigma_3 = 0$ in the direction of increasing μ the unstable focus becomes stable. The saddle index

$$\sigma_4 = (P_\varphi{}' + Q_\rho{}')_4 = \frac{2}{\pi}(1 + \pi - \pi\lambda)$$

vanishes on the straight line $\lambda = 1 + \pi^{-1}$, joining up with the line on which the focus O_3 changes stability, at the point of intersection with the boundary curve $\lambda = (1 - 2\pi^{-1}\varphi_0)\mu + 1$. Starting from the known structures of the phase-space partitioning on the boundary curve (the structures in Fig. 1 from (II-III) to (V)) and in the region $\lambda - \mu + 1 < 0$ (the structure in Fig. 2 (8)), we can trace all the bifurcations and the changes of structure under a monotonic rotation of the field as μ increases.

The qualitative parameter partitioning structure does not differ from the partitioning structure of the original system (1). Structures corresponding to the interior points of the parameter-space partitioning regions are equivalent to the structures in the partitioning regions for system (1). For a system with approximations (5) one singular bifurcation arises. A singular point of the type of a center at the point O_3 corresponds to the points of the curve $\sigma_3 = 0$ on the strip $0 < \varphi < \varphi_0$. As μ increases and the index σ_3 changes sign an unstable limit cycle appears from the boundary of the region filled by closed curves. The partitioning in the neighborhood of the equilibrium states O_3 and O_4, shown in Fig. 7, corresponds to the point of the curve $\sigma_3 = 0$

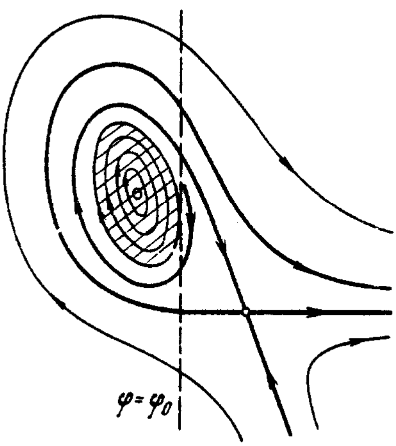

Fig. 7.

Note. The region of closed curves in the neighborhood of the point O_3 cannot have a fused separatrix loop of saddle O_4 as its boundary because the saddle index is nonzero at the saddle. The possibility of realizing the partitioning structure shown in Fig. 7 reveals the nonanalyticity of the right-hand sides of the system under approximations (5).

6. If φ_0 is not small, then under approximations (5), by changing the parameter φ_0 we can alter the behavior of the indices σ_3 and σ_4 so as to eradicate the conditions which have made inevitable the appearance of a region in which two limit cycles girding the cylinder exist. As φ_0 increases upto the value $\varphi_0 = {}^1/_2\pi$ the region in which a separatrix loop arises for a positive value of the saddle index σ_4 vanishes in the space $\mu > 0$, $\lambda > 1$. When $\varphi_0 > {}^1/_2\pi^2(\pi + 1)^{-1}$ the curve $\sigma_3 = 0$ consists of a piece of the hyperbola

$$2\varphi_0\pi\lambda\mu - (2\varphi_0 + \pi^2)\mu + \pi = 0$$

between the straight line $\lambda = 1$ and the point $\lambda = \mu = {}^1/_2\pi\varphi_0{}^{-1}$ at the break in the boundary curve. The curve $\sigma_4 = 0$ consists of a piece of this same hyperbola within the interval ${}^1/_2\pi\varphi_0{}^{-1} < \lambda < 1 + \pi^{-1}$ (saddle O_4 in the interval $0 < \varphi < \varphi_0$) and of the halfline $\lambda = 1 + \pi^{-1}$, $\mu > (\pi - 2\varphi_0)^{-1}$ abutting it (a saddle, in the interval $\varphi_0 < \varphi < {}^1/_2\pi$). As $\varphi_0 \to {}^1/_2\pi$ the boundary curve becomes a polygonal line $\lambda = 1$ $(0 < \mu < 1)$, $\lambda = \mu$ $(\mu > 1)$; the curve $\sigma_3 = 0$ leaves the boundary of the region $\lambda > 1$, being considered, while the curve $\sigma_4 = 0$ turns into a branch of the hyperbola $\pi\lambda\mu - \pi\mu - \mu + 1 = 0$ and, consequently, coincides with the curve $\sigma_3 = 0$, obtained under approximations (2). The parameter-space partitioning is qualitatively equivalent to the partitioning under approximations (2).

Appendix 1. The ω-separatrix of the saddle-node enters the singular point along the direction

$$\varkappa_2 = -2\mu^2 / \lambda = 2(1 - \lambda^2) / \lambda$$

Its tangent at the singular point (φ_0, ρ_0) is

$$\rho - \rho_0 = \varkappa_2(\varphi - \varphi_0) \quad (\varphi_0 = \arcsin \lambda^{-1}, \ \rho_0 = \rho^\circ$$

The tangent intersects the ρ axis at a point with the ordinate

$$\rho_1 = \frac{\mu}{\lambda} + \frac{2(\lambda^2-1)}{\lambda} \arcsin\frac{1}{\lambda}$$

If the ω-separatrix of the saddle-node falls into the region above the maximum $(\mu + 1)/ / \mu$ of the isocline of horizontal slopes, then obviously limit cycles girding the cylinder cannot exist. This is realized automatically for parameter values satisfying the inequality $(1 + \mu)/\mu < \rho_1$ and for which the ω-separatrix lies above the tangent on the interval $0 \leqslant \varphi \leqslant \varphi_0$. As $\lambda \to \infty$ we have$(1 + \mu / \mu \to 1$ and $\rho_1 \to 3$ and, consequently, the inequality indicated is fulfilled for sufficiently large λ.

Let us show that for sufficiently large λ the ω separatrix does lie above the tangent. Consider the points of intersection of the tangent and the isocline in the direction $\varkappa_2$. Eliminating ρ and replacing ρ_0 and $\varkappa_2$ by their values, we arrive at the equation

$$[1-2\mu(\varphi - \varphi_0)] [1 + 2\mu^3(\varphi - \varphi_0) - \lambda \sin\varphi] = \mu [\lambda \cos\varphi - \mu + 2\mu^2 (\varphi - \varphi_0)]$$

The left and right-hand sides of this equation, considered as functions of φ, vanish at the point $\varphi = \varphi_0$ and have coinciding first derivatives. The difference between the values of the second derivatives preserves sign on the whole interval $0 \leqslant \varphi \leqslant \varphi_0$ for sufficiently large λ i.e., the isocline and the tangent do not intersect. The isocline lies below the tangent (the quantity ρ_* — the root of the equation defining the ordinate of the point of intersection of the isocline at $\varkappa_2$ with the ρ axis — tends to unity as $\lambda \to \infty$ and, consequently, $\rho_* < \rho_1$ for large λ). Since for large λ the ω-separatrix close to the point $\varphi = \varphi_0$ lies above the tangent (this will be shown), while the isocline in the direction $\varkappa_2$ lies below the tangent, and since the isocline and the tangent do not intersect on the interval $0 \leqslant \varphi \leqslant \varphi_0$, it is obvious that the ω-separatrix also cannot intersect the tangent and is situated above the tangent on the whole interval $0 \leqslant \varphi \leqslant \varphi_0$. The indicated arrangement of the separatrix and the tangent close to the point follows from the fact that

$$\lim_{\varphi\to\varphi_0} \frac{d^2\rho}{d\varphi^2} = \frac{\mu (8\mu^4 - 5\mu^2 - 1)}{\lambda (2\mu^2 - 1)}, \quad \lim_{\varphi\to\varphi_0} \frac{d\rho}{d\varphi} = \varkappa_2$$

and, consequently, close to the point $\varphi = \varphi_0$ on the ω-separatrix $d^2\rho / d\varphi^2 > 0$ for large λ

Appendix 2. The proof is similar to that presented in Appendix 1. The ω-separatrix of the saddle-node enters the singular point along the direction $\varkappa_2 = -{}^1/_2(4\mu^2 - 1)\mu^{-1}$ and for $\mu \gg 1$ is located above the tangent on the interval $0 \leqslant \varphi \leqslant \varphi_3$

because here

$$\lim_{\varphi\to\varphi_3}\frac{d^2\rho}{d\varphi^2}=\frac{(4\pi^2-1)\,[\pi(4\mu^2-1)-4(\pi+1)]}{\pi\,[\pi(4\mu^2-1)-4]}>0,\qquad \text{if}\quad \lim_{\varphi\to\varphi_3}\frac{d\rho}{d\varphi}=\varkappa_2$$

2. The isocline in direction $\varkappa_2$ does not intersect the tangent on the interval $0<\varphi<\varphi_3$ The equation

$$[1-2\mu(\varphi-\varphi_3)]\left[\frac{1}{2\mu}+\frac{4\mu^2-1}{2}(\varphi-\varphi_3)-\frac{2}{\pi}\varphi\right]=\frac{1}{4\mu}+\frac{4\mu^2-1}{2}(\varphi-\varphi_3)-\frac{4\mu}{\pi^2}\varphi^2$$

has no roots; this is proved analogously to Appendix 1.

3. The ordinate of the point of intersection of the $\varkappa_2$-isocline with the ρ-axis is determined from the equation $4\mu^2(\rho-1)^2=1$ and is less than the corresponding ordinate $\rho_1=(1-{}^1/_4\,\mu^{-2})(1+{}^1/_2\,\pi)$ of the tangent. The limit of the value of ρ_1 as $\mu\to\infty$ is $1+{}^1/_2\,\pi$ and, consequently, for large μ the ω separatrix falls into the region above the maximum of the isocline of horizontal slopes, $\rho_{max}=(\lambda+1)\mu^{-1}=(1+{}^1/_2\mu^{-1})^2$.

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